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Linearized Supersonic Aerofoil Theory. Part I

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LINEARIZED SUPERSONIC AEROFOIL THEORY. PARTS I AND II

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CONTENTS

PART I

Linearized supersonic aerofoil theory is developed by operational methods. It is shown that a wide variety of problems can be handled by these methods, which have the advantage of very directly exhibiting the analogies between supersonic aerofoil and other wave problems. Results for the lift and drag on semi-infinite rectangular wings obtained by the cone-field method of Busemann are confirmed, and a recurrence method is developed for dealing with a finite rectangular wing of arbitrary chord. A very general Green's function method, analogous to that employed in diffraction problems, is also developed by means of which a wide class of problems involving tapered wings or curved leading edges can be solved.

Introduction

The linearized theory for supersonic flow past an aerofoil of infinite aspect ratio was first put forward by Ackeret, and subsequently somewhat extended by Busemann. A summary and critique of this type of theory has recently been given by Lighthill (1944*a*), which shows that the results given by the theory are fairly accurate for moderate Mach numbers and

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generally give a reasonable indication of the resultant forces to be expected on an aerofoil surface. This paper is concerned with the extension of the linearized supersonic theory to flow past finite aerofoils, so far as possible of arbitrary plan and cross-section. In fact, certain limitations will be found in the applicability of the theory, which will be discussed as they arise.

A brief discussion of previous work on supersonic flow past finite aerofoils is perhaps necessary to explain the author's standpoint. The first attempts were made by Schlichting (1936) on the basis of a lifting-surface theory put forward by Prandtl. Unfortunately, an error in the analytical treatment led to the conclusion that the problem could not be solved analytically, and on this basis some false numerical solutions were given. The error in Schlichting's work was first detected by Busemann (1943) who, by the so-called 'cone-field' method, provided an analytical answer to the problem considered by Schlichting—that of the rectangular plate aerofoil at incidence. In this country Lighthill (1944*b*) in the meantime refined the conceptions underlying Schlichting's work, introducing the idea of supersonic 'sources' and 'doublets', analogous to those used by Karman and Tsien. By this means Lighthill showed that no induced drag was to be expected on an aerofoil of rectangular plan—with chord less than a certain limiting length—but did not detect the analytical error in Schlichting's treatment. Taunt & Ward (1946), following up Lighthill's work, succeeded in deriving the correct analytical solutions for flow past some straight-edged aerofoils. More recently, the author has had the benefit of seeing an advance copy of some further solutions by Ward of problems concerned with straight-edge aerofoils treated by a modification of Busemann's cone-field method.

The author's initial approach to the problem was made on the hypothesis that lifting surface, or, what is the same thing, supersonic doublet, methods were unlikely to be the best means of handling the problem. The equation for the velocity potential in linearized supersonic flow is exactly the two-dimensional wave equation, and suggests to any problem of flow past an aerofoil many physical and formal analogues in the field of acoustics, hydrodynamics and electromagnetic vibrations. It was felt that the methods most fruitful in the latter type of problem would also be most suitable for the treatment of the present aerofoil problems. An additional advantage of this approach was the hope that the physical analogies might lead to a better physical insight into the aerofoil problem, if not suggest an experimental analogue capable of more simple measurement. This, then, must be the excuse for bringing yet another method to bear on the linearized aerofoil problem.

Fundamental equations

It will generally be assumed that the aerofoil is at rest in a supersonic stream with undisturbed velocity U at infinity. The direction of the stream is taken as along the z-axis, and the aerofoil is assumed to lie, to the first order, in the plane $y=0$. The x-axis is then along the span of the aerofoil.

We assume that the deviations of the velocity components, pressure and density from their steady values are infinitesimally small, so that their squares can be neglected. The components of the velocity disturbance will be denoted by (u, v, w). Using the Euler equations, and the continuity equation, together with the equation of state for the gas, assumed perfect, and the adiabatic condition, we can easily find the equation which must be satisfied by the velocity potential from which (u, v, w) can be derived. In fact, for steady motion we have

$$\left.\begin{aligned} U\frac{\partial u}{\partial z} = -\frac{1}{\rho_1}\frac{\partial p}{\partial x}, \quad U\frac{\partial v}{\partial z} = -\frac{1}{\rho_1}\frac{\partial p}{\partial y}, \quad U\frac{\partial w}{\partial z} = -\frac{1}{\rho_1}\frac{\partial p}{\partial z}, \\ \rho_1\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+U\frac{\partial \rho}{\partial z} = 0, \\ p/\rho = RT, \quad p/\rho^\gamma = \text{const.}, \end{aligned}\right\} \tag{1}$$

where ρ_1 is the undisturbed density of the fluid. If we let

$$(u, v, w) = U \operatorname{grad} \phi,$$

it is easily seen from the above equations that, to the first order,

$$\nabla^2\phi = \frac{U^2}{a_1^2}\frac{\partial^2\phi}{\partial z^2}, \tag{2}$$

where a_1 = velocity of sound in the undisturbed medium. Also the excess pressure due to the velocity disturbance is found from the expression

$$\Delta p = -\rho_1 Uw = -\rho_1 U^2 \frac{\partial \phi}{\partial z}. \tag{3}$$

As usual, we shall denote the incident Mach number U/a_1 by M. In the supersonic cases treated $M > 1$, and we introduce the symbol α such that

$$(M^2 - 1) = \alpha^2.$$

The Mach angle μ is then $\cot^{-1}\alpha$.

We have then to deal with the equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = \alpha^2\frac{\partial^2\phi}{\partial z^2}, \tag{4}$$

subject in each case to the appropriate boundary conditions.

Boundary conditions

At any point on the surface of the aerofoil the flow of the fluid must be along the surface. Suppose, in particular, that one of the aerofoil surfaces is given by $y = \eta(x, z)$, where η is a small quantity. Then, it can at once be seen that the condition for tangential flow, at any point on the surface, is that, at that point,

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial z}.$$

The further assumption is now made, consistent with our assumption of a first-order theory, that the above value of $\partial\phi/\partial y$ is applied, not at the point on the aerofoil surface, but at its projection on the plane $y = 0$. Accordingly, if the upper and lower surfaces of the aerofoil are given by

$$y = \eta_1(x, z), \quad y = \eta_2(x, z)$$

respectively, then the boundary condition at the aerofoil surface is that

$$\frac{\partial\phi}{\partial y}\rightarrow\frac{\partial\eta_1(x,z)}{\partial z} \quad \text{or} \quad \frac{\partial\eta_2(x,z)}{\partial z}$$

over the plan of the aerofoil, respectively, as $y\rightarrow 0$ from above and below. This, together with the conditions at infinity and the condition that there shall be no propagation of disturbance upstream, is sufficient to determine ϕ uniquely. If the origin O is taken at a point on the leading edge upstream of which there will be no disturbance, then the latter condition can be mathematically contained in the statement that, over the plane $z=0$, $\phi, \dfrac{\partial\phi}{\partial z}=0$.

It is found convenient to divide the actual problems encountered, according to their boundary conditions, into two different classes—the symmetrical and the anti-symmetrical cases. Thus, for the general case referred to above, we may write

$$2\frac{\partial}{\partial y}\phi(x,+0,z)\left(=2\frac{\partial\phi}{\partial y}\right)_{y=0+}=\frac{\partial}{\partial z}\{\eta_1(x,z)+\eta_2(x,z)\}+\frac{\partial}{\partial z}\{\eta_1(x,z)-\eta_2(x,z)\},$$

$$2\frac{\partial}{\partial y}\phi(x,-0,z)=\frac{\partial}{\partial z}\{\eta_1(x,z)+\eta_2(x,z)\}-\frac{\partial}{\partial z}\{\eta_1(x,z)-\eta_2(x,z)\},$$

where, by an obvious notation $\phi(x,+0,z)$ denotes the value of ϕ at a point on the xz plane as $y\rightarrow 0$ through positive values. It will thus be sufficient if we can solve the two following problems, which we classify according to the symmetry of ϕ:

Symmetrical problem—boundary conditions:

$$\left(\frac{\partial\phi}{\partial y}\right)_{y=0+}=\frac{\partial\eta(x,z)}{\partial z}=f(x,z) \quad \text{say,} \quad \left(\frac{\partial\phi}{\partial y}\right)_{y=0-}=-\frac{\partial\eta(x,z)}{\partial z}=-f(x,z). \tag{5}$$

Anti-symmetrical problem:

$$\left(\frac{\partial\phi}{\partial y}\right)_{y=0+}=\frac{\partial\eta(x,z)}{\partial z}=g(x,z) \quad \text{say,} \quad \left(\frac{\partial\phi}{\partial y}\right)_{y=0-}=+\frac{\partial\eta(x,z)}{\partial z}=g(x,z) \tag{6}$$

The first of these problems corresponds, in practice, with the case of a symmetrical aerofoil at zero incidence. The second problem corresponds with the case of a thin plate aerofoil which may be at any incidence. In the particular case of a flat plate aerofoil $g(x,z)$ becomes the constant ϵ over the plane of the aerofoil, where ϵ is the angle of incidence.

Operational transform of problem

With axes as defined previously we introduce the Laplace transform, with respect to z, of the fundamental equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\alpha^2\frac{\partial^2\phi}{\partial z^2}.$$

Thus, if $\bar{\phi}(x,y)=\int_0^\infty e^{-pz}\phi(x,y,z)\,dz$ is the Laplace transform of ϕ, then, using the fact that with our choice of axes ϕ, $\partial\phi/\partial z$ are zero over the plane $z=0$

$$\frac{\partial^2\bar{\phi}}{\partial x^2}+\frac{\partial^2\bar{\phi}}{\partial y^2}=\alpha^2p^2\bar{\phi}. \tag{7}$$

The method can only be used effectively if a boundary condition for the transform equation can be found, corresponding to the condition for $\partial\phi/\partial y$ over the surface of the aerofoil. In fact

$$\frac{\partial\bar{\phi}}{\partial y} = \int_0^\infty e^{-pz}\left(\frac{\partial\phi}{\partial y}\right)dz,$$

so that the value of $\partial\phi/\partial y$ must be known for a given x and y for all values of z before $\partial\bar{\phi}/\partial y$ can be found. This at once enables us to distinguish the problems which can be approached directly by this operational method. In the first place all problems with boundary conditions of the symmetrical type can be so approached, for in these cases it is clear that, except over the plan of the aerofoil, $\partial\phi/\partial y$ is zero everywhere over the plane $y = 0$.

For the anti-symmetrical problems more rigid limits are set. The leading edge need not be normal to the incident stream, nor indeed straight, but it must nowhere be 'swept back' at as much as the Mach angle. By this condition it is ensured that ahead of the aerofoil $\partial\phi/\partial y = 0$ over the plane $y = 0$. If the aerofoil is finite then its ends must be parallel to the incident stream. In the theoretical treatment it may generally be supposed that the aerofoil extends an infinite distance downstream. In practice for an aerofoil of finite chord the most important information sought is the value of the excess pressure at any point on its surface. Provided that the trailing edge is also at no point swept back beyond the Mach angle, this pressure will be the same as that on an infinite aerofoil continued in any way beyond the trailing edge. The flow behind the trailing edge may then be investigated by the usual methods. Effectively, this involves simply the solution of the two-dimensional wave equation subject to certain initial conditions of velocity and displacement. As the problem does not appear to be of special interest it will not be discussed here.

Analogous physical problems

Before proceeding specifically to the solution of the aerofoil problem, it may help both for the better physical understanding of the question, and to suggest methods of approach, to consider some analogous physical problems. The three most obvious analogies are: (*a*) long gravity waves on a plane sheet of water of uniform depth, (*b*) cylindrical waves of sound, (*c*) transverse vibrations of a uniformly stretched membrane. Of these we shall only refer to the first in any detail; the fundamental equations for this case are recalled briefly below.

We suppose we have a plane sheet of water of uniform depth h. Then the equation for the velocity potential ϕ in the wave propagation is

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}, \quad c^2 = gh, \tag{8}$$

where the x-, y-axes are taken in the surface plane of the water. The elevation of the surface at any point above its mean level is given by

$$\zeta = -\frac{h}{c^2}\frac{\partial\phi}{\partial t}. \tag{9}$$

Comparing with the aerofoil problem we see the two problems are formally identical if, in the former, we replace z by t and α by $1/c$. To any aerofoil problem with given boundary conditions will correspond a wave problem, and from (3) and (9) we see that elevation of the surface in the wave problem corresponds to the excess pressure in the aerofoil problem. The 'wave drag' in the aerofoil problem simply corresponds, when multiplied by the stream velocity, to the energy of the wave system in the long-wave case.

The analogy is particularly simple for the case of the finite flat plate rectangular aerofoil at incidence. Here the wave analogy is simply the case of a plank, of length corresponding to the aerofoil span, moved with appropriate velocity normal to its length.

Finite symmetrical aerofoil at zero incidence

As a first example of the methods to be employed we consider the simple case of an aerofoil of rectangular plan, with span $2b$ and chord c, the leading edge extending along the x-axis from $x=-b$ to $x=+b$. The surface of the aerofoil we assume to be the same for all x, given, say, by

$$y=\eta(z).$$

Then as boundary condition we have

$$\frac{\partial\phi}{\partial y}=\begin{cases}\eta'(z) & y=0+,\\ -\eta'(z) & y=0-,\end{cases}\quad \text{for}\quad \begin{matrix}-b<x<b,\\ 0<z<c.\end{matrix}$$

For this symmetrical type of problem it is, as noted above, clear that $\partial\phi/\partial y=0$ off the aerofoil on the plane $y=0$.

So $$\left(\frac{\partial\bar{\phi}}{\partial y}\right)_{y=0}=\int_0^\infty e^{-pz}\left(\frac{\partial\phi}{\partial y}\right)_{y=0}dz=\int_0^c e^{-pz}\eta'(z)\,dz=f(p)\ \text{say}\quad\text{for}\quad -b\leqslant x\leqslant b$$

and $$\left(\frac{\partial\bar{\phi}}{\partial y}\right)_{y=0}=0\quad(|x|>b).$$

The mathematical problem is thus a fairly simple one. We are given that $\bar{\phi}$ satisfies the equation

$$\frac{\partial^2\bar{\phi}}{\partial x^2}+\frac{\partial^2\bar{\phi}}{\partial y^2}=\alpha^2p^2\bar{\phi},$$

and along the x-axis $\partial\bar{\phi}/\partial y=0$, except for the cut from $x=-b$ to $x=+b$. Along this cut

$$\frac{\partial\bar{\phi}}{\partial y}\to\pm f(p)\quad\text{as}\quad y\to\pm 0. \tag{10}$$

Further, $\bar{\phi}\to 0$ as $y\to\pm\infty$, if the disturbance is only a finite one.

It is immediately clear that $\bar{\phi}$ is an even function of y, thus only positive values of y will be considered. $\bar{\phi}$ is also an even function of x, so that

$$\cos\alpha qx\,e^{-\alpha\sqrt{(p^2+q^2)}\,y}\quad(y\geqslant 0) \tag{11}$$

forms a suitable elementary solution, out of which to build the complete solution to our problem. We assume

$$\bar{\phi}=\int_0^\infty a(q)\cos\alpha qx\,e^{-\alpha\sqrt{(p^2+q^2)}\,y}\,dq. \tag{12}$$

Then, granted the possibility of differentiating under the integral sign,

$$\frac{\partial\bar{\phi}}{\partial y}=-\alpha\int_0^\infty\sqrt{(p^2+q^2)}\,a(q)\cos\alpha qx\,e^{-\alpha\sqrt{(p^2+q^2)}\,y}\,dq. \tag{13}$$

The boundary condition (10) enables $a(q)$ to be determined. We find in fact

$$a(q) = -\frac{2}{\pi\sqrt{(p^2+q^2)}}\frac{\sin\alpha qb}{\alpha q}f(p). \tag{14}$$

Hence we have the solution

$$\begin{aligned}\bar{\phi} &= -\frac{2}{\pi\alpha}f(p)\int_0^\infty \frac{\cos\alpha qx\sin\alpha qb}{q\sqrt{(p^2+q^2)}}e^{-\alpha\sqrt{(p^2+q^2)}y}\,dq \\ &= -\frac{2}{\pi\alpha}\int_0^c e^{-p\zeta}\eta'(\zeta)\,d\zeta\int_0^\infty \frac{\cos\alpha qx\sin\alpha qb}{q\sqrt{(p^2+q^2)}}e^{-\alpha\sqrt{(p^2+q^2)}y}\,dq.\end{aligned} \tag{15}$$

The inverse transform for $\dfrac{e^{-p\zeta}\,e^{-\alpha\sqrt{(p^2+q^2)}y}}{\sqrt{(p^2+q^2)}}$ is well known. It is given by

$$\begin{cases} 0 & \text{for} \quad 0<z<\zeta+\alpha y, \\ J_0[q\sqrt{\{(z-\zeta)^2-\alpha^2y^2\}}] & \text{for} \quad z>\zeta+\alpha y. \end{cases}$$

Hence we find, for the case $0<(z-\alpha y)<c$,

$$\phi = -\frac{2}{\pi\alpha}\int_0^{(z-\alpha y)}\eta'(\zeta)\,d\zeta\int_0^\infty J_0[q\sqrt{\{(z-\zeta)^2-\alpha^2y^2\}}]\frac{\cos\alpha qx\sin\alpha qb}{q}\,dq. \tag{16}$$

The second integral is of a well-known type. For our purposes it is most instructive to use the form

$$\left.\begin{aligned}\int_0^\infty \frac{\sin aq\,J_0(bq)}{q}\,dq &= \int_0^a \frac{d\xi}{\sqrt{(b^2-\xi^2)}}, \quad \text{if } a\leqslant b, \\ &= \tfrac{1}{2}\pi, \quad \text{if } a\geqslant b,\end{aligned}\right\} \tag{17}$$

where we suppose both a, $b>0$. Using this integral the final solution for ϕ can easily be deduced. It is that

$$\phi = -\frac{1}{\pi}\alpha\iint_S \frac{\eta'(\zeta)\,d\xi\,d\zeta}{\sqrt{[(z-\zeta)^2-\alpha^2\{y^2+(x-\xi)^2\}]}}, \tag{18}$$

where the area of integration S is that part of the region

$$-(b+x)\leqslant\xi\leqslant(b-x), \quad 0\leqslant\zeta\leqslant z-\alpha y,$$

which satisfies $(z-\zeta)^2\geqslant\alpha^2\{y^2+(x-\xi)^2\}$. Geometrically, treating (ζ,ξ) as co-ordinates of position on the aerofoil surface, this area can be regarded as that portion of the aerofoil surface within the forward Mach cone from the 'field point' (x,y,z). This solution was obtained by other means by Lighthill. He interpreted

$$\frac{1}{\sqrt{[(z-\zeta)^2-\alpha^2\{y^2+(x-\xi)^2\}]}}$$

as the potential of a unit 'supersonic' source situated at the point $(\xi,0,\zeta)$, having regard to the analogous position which this fundamental solution holds with respect to equation (4), compared with the inverse distance solution for the potential equation.

The interpretation of our result is then clear. We see that each element of area $d\xi\,d\zeta$ of the aerofoil acts as a supersonic source of strength $-\dfrac{\eta'(\zeta)\,d\xi\,d\zeta}{\pi\alpha}$. The potential at any field point is then found by integrating over the sources which can contribute to its potential, i.e. those within the forward Mach cone from the field point.

The result has been demonstrated for an aerofoil of rectangular plan, but can be immediately extended to any shape of aerofoil. If the surface of such an aerofoil is, in general, given by

$$y = \eta(x, z),$$

then the supersonic source strength per unit area is

$$-\frac{1}{\pi\alpha}\frac{\partial}{\partial z}\eta(x, z).$$

So far we have been dealing with the case of a symmetrical aerofoil at zero incidence, which is of interest only in giving the magnitude of wave drag for this case. Before proceeding to the detailed consideration of this drag it will be advantageous to consider the extension of our methods, necessary to deal with the antisymmetrical case, from which the lift on a thin aerofoil at incidence may be calculated.

Mathematically the simplicity of the zero incidence case arises from the fact that $\partial\phi/\partial y$ can be specified over the plane in which the aerofoil lies. When we take the Laplace transform, then in the consequent two-dimensional equation for $\bar{\phi}$, $\partial\bar{\phi}/\partial y$ is known along the whole of the x-axis. Thus, when we represent $\bar{\phi}$ by a Fourier integral, the consequent integral equation to determine the coefficients is an easy one, involving only the inversion of the Fourier integral. In the case of incidence circumstances are entirely different, for now we have no knowledge of $\partial\phi/\partial y$ except at the aerofoil surface. To illustrate the difficulties involved we shall consider briefly an important special case—that of a flat plate aerofoil of rectangular plan, set at incidence in a supersonic stream, with its leading edge normal to the stream. For the theoretical treatment we suppose the chord of the aerofoil to be infinite. It is clear that, if the trailing edge is parallel to the leading edge for any actual case, then the pressures on the wing are the same as they would be for an infinitely extended aerofoil.

Finite rectangular plate aerofoil at incidence

Once more we assume the span of the aerofoil to be $2b$, and assume that the leading edge lies along the x-axis from B', where $x = -b$ to B, where $x = +b$. The equation satisfied by $\bar{\phi} = \left(\int_0^\infty e^{-pz}\phi\,dz\right)$ is again

$$\frac{\partial^2\bar{\phi}}{\partial x^2} + \frac{\partial^2\bar{\phi}}{\partial y^2} = \alpha^2 p^2 \bar{\phi},$$

and the boundary condition is that $\partial\bar{\phi}/\partial y$ takes the value $-\epsilon/p$ as $y \to 0$ from above or below, for $-b < x < b$. $\bar{\phi}$ is continuous except across the cut $B'B$, and tends to zero as $y \to \infty$. From symmetry considerations it is clear that $\bar{\phi}$ is an odd function of y, and has the value zero along the x-axis, except in the cut from B' to B. Across this cut there is a discontinuity in $\bar{\phi}$. We assume that for y positive the solution is expressed in the form

$$\bar{\phi} = \int_0^\infty a(q)\, e^{-\alpha\sqrt{(p^2+q^2)}\,y} \cos \alpha q x\, dq,$$

then the integral equation to determine $a(q)$ may be written

$$\int_0^\infty \sqrt{(p^2+q^2)}\, a(q) \cos \alpha q x \, dq = \frac{\epsilon}{p\alpha}, \quad 0<x<b,$$

$$\int_0^\infty a(q) \cos \alpha q x \, dq = 0, \quad x>b.$$

The integral equation is of the type called 'dual' by Titchmarsh, and its direct solution does not appear to present a simple analytical problem.

The other possible direct method of solving this problem would be by the introduction of elliptic co-ordinates. This possibility has been briefly investigated. Whilst we can find a formal solution in terms of the elliptic cylinder wave functions (Mathieu functions), its interpretation, in the light of existing knowledge of Mathieu functions, is of impracticable difficulty. Solutions have been given by Strutt (1932) for analogous problems to this one, but involving only a single frequency, e.g. radiation of simple harmonic sound waves from a vibrating lamina of finite width but of infinite length. For such problems the Mathieu function treatment is quite satisfactory, and leads to numerical results without undue labour.

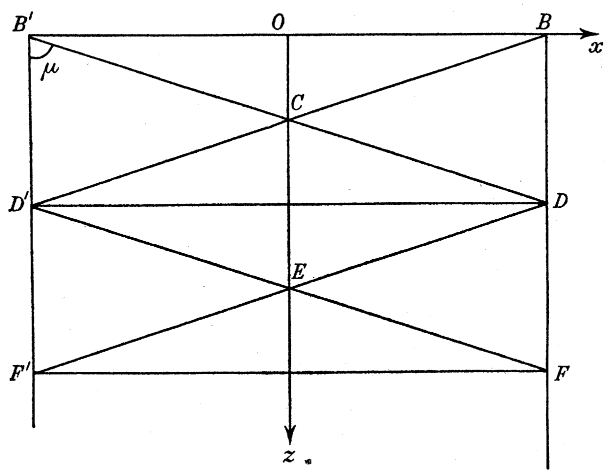

FIGURE 1

The matter is quite different, however, when a transient problem is considered, involving an infinite range of complex frequencies for its operational treatment. Rather a similar situation will be found later when the semi-infinite aerofoil problem is discussed. Here once more the treatment of the boundary-value problem in terms of the appropriate proper function (parabolic cylinder functions) leads only very indirectly to the result, which can be obtained much more simply by other methods. Such simpler methods have not been found for the 'finite-strip' problem, and we are forced to the conclusion that no complete solution of the rectangular aerofoil problem at incidence and with infinite chord is easily obtainable.

Fortunately, however, in practice such a solution is not required. Suppose, for example, that the aerofoil is of chord less than $b \cot \mu (= \alpha b)$. This will generally be true in practice except at Mach numbers just slightly greater than 1. Then the Mach lines from the leading corners BB' (figure 1) will not intersect on the aerofoil, and it is clear that the solution of the problem can be found from that for a semi-infinite wing—the leading edge extending along the x-axis—from 0 to $+\infty$ say. This solution, found by superposing two semi-infinite wing solutions, will still hold after the two Mach lines intersect at C—the two 'linearized' waves radiating from the two edges simply adding. It will cease to hold at DD', when the waves from B, B' first reach the other ends of the aerofoil at D' and D respectively. However, a process of successive approximation will be found by which the solution can be continued, first for the portion DF, and then for successive portions of length $2\alpha b$. The process, in fact, consists in the addition to the original wave system of successively more complicated waves beginning at DD', FF' and so on.

Mathematically, the technique required for this treatment is very similar to that introduced by Sommerfeld (1897) in his classic discussion of the diffraction of electromagnetic waves at a semi-infinite perfectly conducting obstacle, and later simplified and developed by various authors, including Lamb and Carslaw. In particular, our method of the successive introduction of new waves at D, F, etc., is very similar to the approximate method employed by Schwarzschild (1902) for the solution of the problem of diffraction at a slit of monochromatic electromagnetic radiation. In fact, the method is much better adapted for our problem than for Schwarzschild's, for in his case the complete infinite series was required for a solution, and its convergence is sometimes slow, whereas in our case each term of the series gives an exact solution for an aerofoil with chord extended by a further distance.

In this discussion of these problems it may aid description if one of the more picturesque physical analogies is considered in parallel with the aerofoil problem. For this purpose the long-wave case is chosen—the analogous problem being the discussion of the wave system produced by a semi-infinite plank moved normal to itself in a shallow sheet of water.

Semi-infinite wing problem

We suppose the leading edge to extend along the x-axis from $x=0$ to ∞. In the notation of the last paragraph, we require to find the solution of

$$\frac{\partial^2\bar{\phi}}{\partial x^2}+\frac{\partial^2\bar{\phi}}{\partial y^2} = \alpha^2p^2\bar{\phi},$$

making $\partial\bar{\phi}/\partial y \to -\epsilon/p$ as $y\to 0$ from above or below for $x>0$, such that also $\phi\to 0$ as $y\to\pm\infty$. For economy of notation we write $\alpha p = k$. Following Lamb (1932) we assume that a solution of the equation may be obtained in the form

$$\bar{\phi} = e^{-ky}u. \tag{19}$$

The equation for u becomes
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 2k\frac{\partial u}{\partial y}. \tag{20}$$

We now introduce parabolic co-ordinates (ξ,η) defined by the relation

$$(x+iy) = (\xi+i\eta)^2,$$

so that (20) becomes
$$\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}-4k\left\{\eta\frac{\partial u}{\partial \xi}+\xi\frac{\partial u}{\partial \eta}\right\} = 0. \tag{21}$$

This equation has solutions in the form $u=f(\xi+\eta)$ or $u=g(\xi-\eta)$, where f and g satisfy respectively the equations

$$\frac{d^2f}{d\tau^2}\mp 2k\tau\frac{df}{d\tau} = 0, \tag{22}$$

Similar solutions are found in the form $\bar{\phi} = e^{ky}u$, where again $u = f(\xi+\eta) = f(\tau)$ say or $u = g(\xi-\eta)$, the equations for f and g being as before but with the sign of k changed. For the first case, on integrating, we have

$$f = A + B\int^{\tau} e^{k\tau^2}\,d\tau, \quad g = C + D\int^{\tau} e^{-k\tau^2}\,d\tau.$$

Combining the various possible solutions, remembering that $\bar{\phi} \to 0$ as $y \to \pm\infty$, we find as a solution satisfying the boundary conditions

$$\bar{\phi} = -\frac{\epsilon}{p}\frac{1}{\sqrt{(\pi\alpha p)}}\left[e^{\alpha p y}\int_{\xi+\eta}^{\infty} e^{-\alpha p\tau^2}\,d\tau - e^{-\alpha p y}\int_{\eta-\xi}^{\infty} e^{-\alpha p\tau^2}\,d\tau\right]. \tag{23}$$

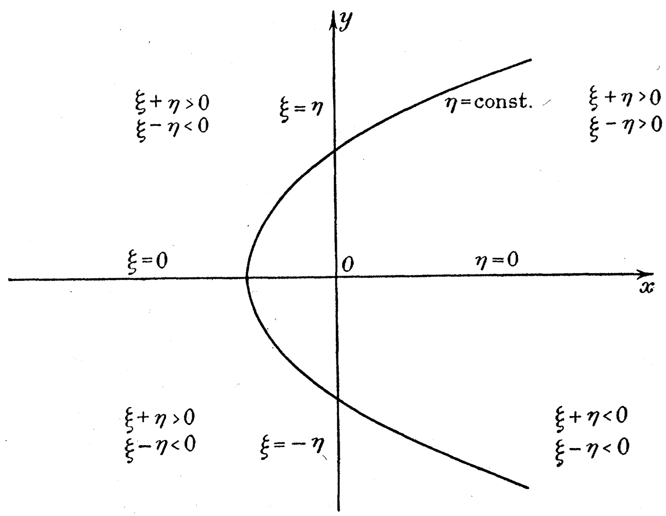

FIGURE 2

This is the unique solution of our problem. The signs of the expressions $\xi \pm \eta$ in the different quadrants must be carefully observed. Their behaviour is summarized in figure 2. $\bar{\phi}$ is, of course, an odd function of y. It is zero along the negative x-axis, and has a discontinuous change of value across the positive x-axis. If we denote by r the distance of any point $P(x,y)$ from O, then

$$\xi+\eta = \pm\sqrt{(r+y)}, \quad \xi-\eta = \pm\sqrt{(r-y)}.$$

$\bar{\phi}$ can then be written for the case where $y>0$ in the more significant form

$$\frac{\bar{\phi}}{\epsilon} = \left\{\frac{1}{\alpha}\frac{e^{-\alpha p y}}{p^2}\right\} - \frac{1}{p\sqrt{(\pi\alpha p)}}\left[e^{\alpha p y}\int_{\sqrt{(r+y)}}^{\infty} e^{-\alpha p\tau^2}\,d\tau \pm e^{-\alpha p y}\int_{\sqrt{(r-y)}}^{\infty} e^{-\alpha p\tau^2}\,d\tau\right], \tag{24}$$

where the first term is only present for $x>0$, and in the second term the $+$ and $-$ signs correspond to $x>0$, and $x<0$ respectively.

To complete the solution we must interpret the value obtained for $\bar{\phi}$ by inverting the Laplace transform. This is easily done, and we find that, for $y>0$,

$$\frac{\phi}{\epsilon} = \frac{1}{\alpha}(z-\alpha y) - \frac{2}{\pi}\left[\int_{\sqrt{(r+y)}}^{\sqrt{\left(\frac{z}{\alpha}+y\right)}} \sqrt{\left\{\left(\frac{z}{\alpha}+y\right)-\tau^2\right\}}\,d\tau \pm \int_{\sqrt{(r-y)}}^{\sqrt{\left(\frac{z}{\alpha}-y\right)}} \sqrt{\left\{\left(\frac{z}{\alpha}-y\right)-\tau^2\right\}}\,d\tau\right], \tag{25}$$

where the first term only occurs for $x>0$, $z>\alpha y$, and the second term only occurs for $z>\alpha r$, having $+$ or $-$ sign according as $x>0$ or $x<0$. The solution can be written

$$\frac{\phi}{\epsilon}=\frac{1}{\alpha}(z-\alpha y)-\frac{1}{\pi}\left[\left\{\left(\frac{z}{\alpha}+y\right)\cos^{-1}\sqrt{\left\{\frac{\alpha(r+y)}{z+\alpha y}\right\}}-\sqrt{(r+y)}\sqrt{\left(\frac{z}{\alpha}-r\right)}\right\}\right.$$
$$\left.\pm\left\{\left(\frac{z}{\alpha}-y\right)\cos^{-1}\sqrt{\left\{\frac{\alpha(r-y)}{z-\alpha y}\right\}}-\sqrt{(r-y)}\sqrt{\left(\frac{z}{\alpha}-r\right)}\right\}\right], \tag{26}$$

with the same provisions as above regarding the presence of terms and their signs. The dimensionless velocity components will also be given for reference. They are, for $y>0$,

$$\frac{1}{\epsilon}\frac{\partial\phi}{\partial x}=\frac{2}{\pi}\sqrt{\left(\frac{z}{\alpha r}-1\right)}\cos\tfrac{1}{2}\theta, \tag{27a}$$

$$\frac{1}{\epsilon}\frac{\partial\phi}{\partial y}=(-1)+\frac{1}{\pi}\left[\sin^{-1}\sqrt{\left\{\frac{\alpha(r+y)}{z+\alpha y}\right\}}\mp\sin^{-1}\sqrt{\left\{\frac{\alpha(r-y)}{z-\alpha y}\right\}}\right]+\frac{2}{\pi}\sqrt{\left(\frac{z}{\alpha r}-1\right)}\sin\tfrac{1}{2}\theta \tag{27b}$$

(first term appears only for $x>0$, $z>\alpha y$, remaining terms appear only for $z>\alpha r$, and the $-$ and $+$ signs occur for $x>0$, $x<0$ respectively)

$$\frac{1}{\epsilon}\frac{\partial\phi}{\partial z}=\left(\frac{1}{\alpha}\right)-\frac{1}{\pi\alpha}\left[\cos^{-1}\sqrt{\left\{\frac{\alpha(r+y)}{z+\alpha y}\right\}}\pm\cos^{-1}\sqrt{\left\{\frac{\alpha(r-y)}{z-\alpha y}\right\}}\right] \tag{27c}$$

(first term appears only for $x>0$, $z>\alpha y$, second term appears only for $z>\alpha r$, and is $+$ or $-$ according as $x>0$, $x<0$).

The physical significance of the various terms is perhaps most quickly grasped when we think of the analogous water-wave problem. Suppose we imagine a barrier along the positive x-axis to be moved with velocity ϵ in the negative y-direction. The solution for the velocity potential for this problem is identical with that for the aerofoil problem, provided only we replace z by the time t, and $1/\alpha$ by the velocity of propagation c. This wave problem is perhaps best described by the surface elevation ζ, for which we have the expression ($y>0$)

$$\zeta=-\frac{h\epsilon}{c}\left[1-\frac{1}{\pi}\left\{\cos^{-1}\sqrt{\left(\frac{r+y}{ct+y}\right)}\pm\cos^{-1}\sqrt{\left(\frac{r-y}{ct-y}\right)}\right\}\right],$$

with the obvious cautions as to the appearance of the various terms. The first term provides what we may term the 'plane-wave' contribution to the solution. It is present only for $x>0$, $y<ct$ and thus represents a box-like region of constant elevation ($=-h\epsilon/c$) spreading out from the barrier with velocity c. The second term represents a somewhat complex circular wave, at any time lying within the circle centre O of radius ct (figure 3). This wave arises from the flow around the corner O, which tries to equalize the elevation of the free surface on the two sides of the barrier, just as in the aerofoil case flow occurs from the high- to the low-pressure side of the aerofoil. Diagrams of the surface level along lines $A'OA$, $B'OB$ are given below (figure 4). The disturbance within the circular wave spreading out from O with increasing time corresponds exactly, of course, to the disturbance in the aerofoil case spreading from the leading corner within its Mach cone. The diagram may be taken to depict either a section of the aerofoil problem at some given value of z by a plane perpendicular to the z-axis, just as it shows the elevation in the wave problem at a given time.

The quantity of principal interest is the excess pressure on the aerofoil surface. From (27) we see that this is given by

$$\frac{p}{\rho U^2} = -\frac{2\epsilon}{\pi\alpha}\sin^{-1}\sqrt{\left(\frac{\alpha x}{z}\right)} \tag{28}$$

for points between the Mach line from O and the edge of the aerofoil. Elsewhere, on the wing the excess pressure is as in the two-dimensional problem. All the formulae for the velocity components are, naturally, homogeneous functions of the co-ordinates of the point of measurement, as from the physical nature of the problem they can depend only on the angular position relative to O. This fact is used as the basis of the cone field of Busemann developed by Ward, to the first of whom the result (28) is originally due.

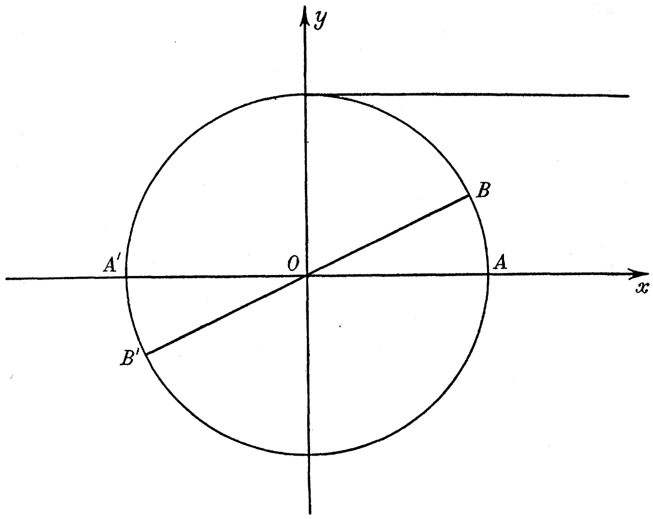

Figure 3

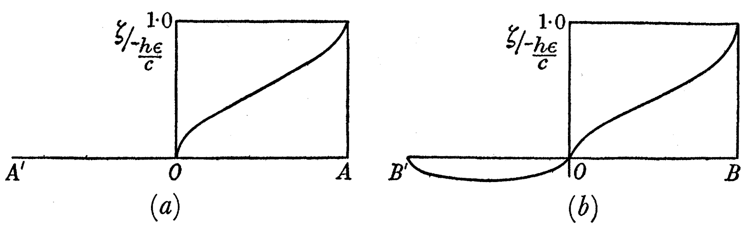

Figure 4

Extension to case of finite wing

The method of extension to the case of a wing of finite span is now evident. Suppose in figure 5 that OO' represent the projections on the xy plane of the two ends of the aerofoil (or in the wave problem, the two ends of a finite barrier), then it is clear that the solution of the problem in which $\partial\bar{\phi}/\partial y$ must take a specified constant value along OO' is given by the superposition of (a) the plane wave $OAA'O'$, (b) two circular waves radiating from O and O', exactly as given in the previous paragraph. This solution will hold (in the wave problem) only up to such a time as the wave from O runs off the other end of the barrier at O', for if allowed to continue unaltered, this wave would produce a discontinuity in the water level across $O'x$, which is physically impossible. In the aerofoil problem it would, equally, produce a discontinuity in the excess pressure in the free space beyond either edge of the aerofoil, for values of z greater than that at which the Mach lines from the leading corners reach the opposite sides (i.e. $z = 2b\alpha$).

We shall be able to continue the solution for a further time $2b/c$ (or for a further length $2\alpha b$) if we can discover a wave which, starting to radiate from O' at time $t = 2b/c$, will annul the discontinuity in potential produced after that time along $O'x$ by the first wave from O. Symbolically we might denote the plane wave by P, and the initial circular waves from O, O' by C_1, C'_1. Then up to time $t = 2b/c$ the solution is given by

$$\phi = P + C_1 + C'_1.$$

Thereafter, C_1 produces a discontinuity along $O'x$, to annul which a wave C'_2 beginning a time $t = 2b/c$ is required. The requirement on C'_2 is that it should produce the required discontinuity across $O'x$, but leave $\partial\phi/\partial y$ unaltered over $O'x'$. Until $t = 4b/c$ the solution is, then, $\phi = P + C_1 + C'_1 + C_2 + C'_2$. At $t = 4b/c$ the wave C'_2 in turn runs off the barrier at O, producing a new discontinuity in ϕ which must be eliminated by waves C_3, C'_3, and so the process may be continued.

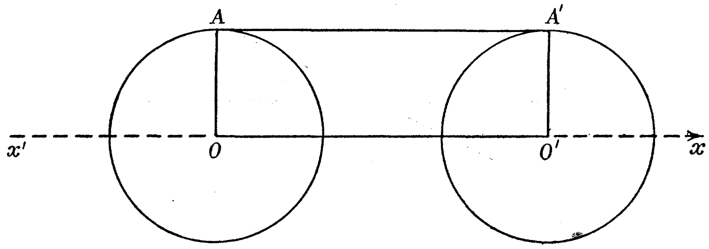

FIGURE 5

For the formal carrying out of this process we must be able to solve the following mathematical problem. Given that

$$\frac{\partial^2\bar{\phi}}{\partial x^2} + \frac{\partial^2\bar{\phi}}{\partial y^2} = k^2\bar{\phi}, \tag{29}$$

to find $\bar{\phi}$ to satisfy the following conditions:

(i) $\bar{\phi}$ is continuous with continuous first derivatives everywhere except along the positive x-axis.

(ii) $\bar{\phi}$ vanishes at $y = \pm\infty$.

(iii) Along the x-axis $\bar{\phi}$ takes certain prescribed values, given by $\bar{\phi} = f(x)$ say, whether we approach the x-axis from above or below.

There are, in fact, four problems of this type which can be solved, the solutions of which will be useful in the sequel. They are given by variation of the condition (iii). Thus

(a) $\bar{\phi} \to f(x)$ as $y \to 0$ from above or below.

(b) $\bar{\phi} \to +f(x)$ as $y \to 0$ from above,
$\qquad -f(x)$ as $y \to 0$ from below.

(c) $\left(\dfrac{\partial\bar{\phi}}{\partial y}\right) \to g(x)$ as $y \to 0$ from above or below.

(d) $\left(\dfrac{\partial\bar{\phi}}{\partial y}\right) \to +g(x)$ as $y \to 0$ from above,
$\qquad -g(x)$ as $y \to 0$ from below.

These problems will all be solved if we can find the Green's functions for (29) with the boundary Ox. From Green's theorem it is known that if u is a solution of the equation $\partial^2u/\partial x^2+\partial^2u/\partial y^2 = k^2u$ continuous with its derivatives inside a plane curve C, and if G is another solution of this equation having a singularity at a point P within C, such that near P, $G \sim -\log\rho$, where ρ denotes distance from P, then $u(P)$ can be expressed in the form

$$u(P) = -\frac{1}{2\pi}\int_C \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\right) ds, \tag{30}$$

where n denotes the direction of the indrawn normal to C. For our problem we require two special choices of G:

$$\left.\begin{aligned} &(a)\quad G_1 = 0 \text{ along } C, \text{ then } u(P) = +\frac{1}{2\pi}\int_C u\frac{\partial G_1}{\partial n} ds,\\ &(b)\quad \frac{\partial G_2}{\partial n} = 0 \text{ along } C, \text{ then } u(P) = -\frac{1}{2\pi}\int_C G_2\frac{\partial u}{\partial n} ds.\end{aligned}\right\} \tag{31}$$

In terms of the functions G_1, G_2 our four boundary-value problems may be solved, the curve C being taken as the x-axis described twice together with the circle at infinity.

Determination of Green's functions for a semi-infinite barrier

Formally, the problem of the determination of the Green's functions for (29) is practically equivalent with the solution of the problem of an infinite line-source of sound waves in the presence of a semi-infinite thin plane. The problem has been treated by Carslaw (1899) in some detail, using Sommerfeld's method for the derivation of multiform solutions of the wave equation, and we shall merely sketch the results in our notation.

Let the point at which $u \sim -\log\rho$ be the point P_0, with polar co-ordinates (r_0, θ_0) relative to the end of the semi-infinite barrier, the barrier being given by $\theta = 0$, and $\theta = 2\pi$. Then $\rho = \surd\{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)\}$ denotes the distance from P_0 to an arbitrary field point P. Now

$$u_0 = K_0(k\rho)$$

is certainly a solution of equation (29), with the appropriate singularity at P_0, K_0 representing the usual modified Bessel function of zero order. This solution can be written, with the help of Cauchy's theorem, as an integral round a contour in the ζ-plane,

$$u_0 = \frac{1}{2\pi}\int_{C(\theta_0)} K_0[k\surd\{r^2+r_0^2-2rr_0\cos(\zeta-\theta)\}]\frac{e^{i\zeta}}{e^{i\zeta}-e^{i\theta_0}}d\zeta, \tag{32}$$

the contour $C(\theta_0)$ being a small circle around the point $\zeta = \theta_0$. In this expression the integrand has singularities given by $\zeta = \theta_0+2m\pi$, $\zeta = \theta \pm i\alpha_1+2m\pi$, m any integer, the latter being branch points. Here we have written

$$\cosh\alpha_1 = \frac{r^2+r_0^2}{2rr_0}.$$

If we denote $\surd\{r^2+r_0^2-2rr_0\cos(\zeta-\theta)\}$ by R, then in deforming the contour $C(\theta_0)$ to ∞ we must take care that the real part of R remains positive. This will be so on the contour shown in figure 6. If the straight parts of the contour shown dotted are separated by 2π, then the integrals over these cancel each other, and we are left with

$$u_0 = \frac{1}{2\pi}\int_{C(\theta)} K_0(kR)\,\frac{e^{i\zeta}\,d\zeta}{e^{i\zeta}-e^{i\theta_0}}, \tag{33}$$

where $C(\theta)$ denotes the curved contour shown. This solution is of period 2π, but a solution of period 4π can at once be formed, namely,

$$u(r,\theta;r_0,\theta_0) = \frac{1}{4\pi}\int_{C(\theta)} K_0(kR)\,\frac{e^{\frac{1}{2}i\zeta}}{e^{\frac{1}{2}i\zeta}-e^{\frac{1}{2}i\theta_0}}\,d\zeta, \tag{34}$$

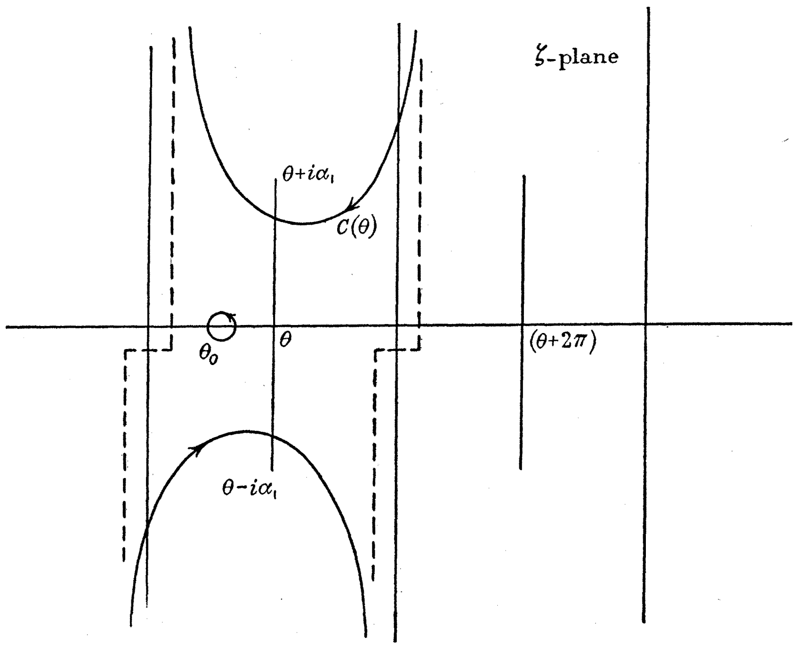

FIGURE 6

the integral being taken over the path $C(\theta)$ appropriate to the value of θ. The properties of u may be summarized as follows:

(i) u satisfies the equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2} = k^2 u$.

(ii) It is periodic in θ with period 4π.

(iii) It is finite and continuous for all values of (r,θ) except at the point $(r_0\theta_0)$, where $u \sim -\log\rho$.

(iv) If the values of u for θ, $\theta+2\pi$ are denoted by u_1, u_2 respectively, then $u_1+u_2 = u_0$.

The required Green's functions G_1, G_2, satisfying the conditions (31), are easily expressed in terms of u, viz.

$$G_1 = u(r,\theta;r_0,\theta_0)-u(r,\theta;r_0,-\theta_0), \quad G_2 = u(r,\theta;r_0,\theta_0)+u(r,\theta;r_0,-\theta_0), \tag{35}$$

where the physical space is given by $0<\theta<2\pi$. The boundary conditions are thus satisfied by an extension of the image method—the essence of the treatment being that the multiform solution allows the image to be removed from the physical plane. G_1, G_2 can be evaluated by choosing as path for u_2 the lines $\zeta = \theta+\pi, \theta+3\pi$, from which u_1 can then be deduced from the relation $u_1 = u_0 - u_2$. The functions found are as follows:

G_1 $\qquad (G_1 = 0 \text{ when } \theta = 0,\ 2\pi)$.

We define three regions

$$0<\theta<\pi-\theta_0, \tag{A}$$

$$\pi-\theta_0<\theta<\pi+\theta_0, \tag{B}$$

$$\pi+\theta_0<\theta<2\pi. \tag{C}$$

Then in these regions G_1 has the form

$$\text{(A)}\quad G_1 = K_0[k\surd\{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)\}]-\frac{1}{\pi}\cos\tfrac{1}{2}(\theta-\theta_0)\int_0^\infty K_0\{k\surd(r^2+r_0^2+2rr_0\cosh b)\}$$
$$\times\frac{\cosh\frac{1}{2}b\,db}{\cos(\theta-\theta_0)+\cosh b}-K_0[k\surd\{r^2+r_0^2-2rr_0\cos(\theta+\theta_0)\}]+\frac{1}{\pi}\cos\tfrac{1}{2}(\theta+\theta_0)$$
$$\times\int_0^\infty K_0[k\surd(r^2+r_0^2+2rr_0\cosh b)]\frac{\cosh\frac{1}{2}b\,db}{\cos(\theta+\theta_0)+\cosh b}. \tag{36}$$

(B) G_1 = as in (A) but without third term.
(C) G_1 = as in (A) but without first and third terms.

G_2 $\qquad \left(\frac{1}{r}\frac{\partial G_2}{\partial\theta}\right) = 0$ when $\theta = 0, 2\pi$.

G_2 is as G_1 but with the signs of the third and fourth terms changed.

With a knowledge of G_1 and G_2 it is possible, in principle using (31), to find a solution $\bar{\phi}$ of (29) such that either $\bar{\phi}$ or its normal derivative takes any prescribed values along the x-axis. A general approach to a wide variety of aerofoil problems is thereby permitted. Before turning to some of these we shall complete the investigation of the rectangular plate at incidence.

Pressure on finite rectangular plate at incidence

We now resume the investigation of a rectangular plate aerofoil of span $2b$, set at incidence ϵ in a supersonic stream. We have already seen how, provided the chord of the aerofoil is less than $2\alpha b[\alpha = \surd(M^2-1)]$, the excess pressure can be found by superposing a plane wave together with two waves, earlier denoted by C_1, C_1' from the leading corners O, O' and confined respectively within the Mach cones from these points. The aim now is to extend the treatment to greater lengths of aerofoil.

Suppose we consider first the wave C_1 radiating from O in figure 7, with velocity potential ϕ_1, say, then ϕ_1 has the transform

$$\bar{\phi}_1 = \frac{1}{\surd(\alpha\pi)}\frac{1}{p^{\frac{3}{2}}}\left\{e^{\alpha py}\int_{\surd(r+y)}^\infty e^{-\alpha p\tau^2}\,d\tau+e^{-\alpha py}\int_{\surd(r-y)}^\infty e^{-\alpha p\tau^2}\,d\tau\right\}. \tag{37}$$

Denoting $x' = (x-2b)$ we see that along $O'x$ (i.e. $y = 0$) $\bar{\phi}_1$ has the value

$$\bar{\phi}_1 = \frac{2}{\surd(\alpha\pi)}\frac{1}{p^{\frac{3}{2}}}\int_{\surd(x'+2b)}^\infty e^{-\alpha p\tau^2}\,d\tau = f(x') \quad \text{say}, \tag{38}$$

where $x'\geqslant 0$.

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The transform for the wave C_2' radiating from A' must cancel this value of $\bar{\phi}_1$ along $O'x$, which, as we have seen earlier, if uncancelled, gives rise to a discontinuity of potential in the triangle $B'A'E$. The form of $\bar{\phi}_2'$ is thus easily found by the Green's function method given above, bearing in mind the further condition that

$$\frac{\partial \bar{\phi}_2'}{\partial y} = 0, \quad y = 0, \quad x' < 0.$$

If n_1, n_2 are used to denote the positive and negative y directions respectively, then, at the field point P_0' with co-ordinates r_0', θ_0' relative to O', we find

$$\bar{\phi}_2'(r_0', \theta_0') = \frac{1}{2\pi}\int_0^\infty f(x')\left\{\frac{\partial}{\partial n_1} G_1(x', 0; r_0', \theta_0') + \frac{\partial}{\partial n_2} G_1(x', 2\pi; r_0', \theta_0')\right\} dx, \tag{39}$$

where G_1 is defined in (36) and we set $k = \alpha p$. Thus we can write

$$\bar{\phi}_2'(r_0', \theta_0') = \frac{1}{2\pi}\int_0^\infty \frac{f(x')}{x'}\frac{\partial}{\partial \theta'}\{G_1(x', 0; r_0', \theta_0') - G_1(x', 2\pi; r_0', \theta_0')\} dx'. \tag{40}$$

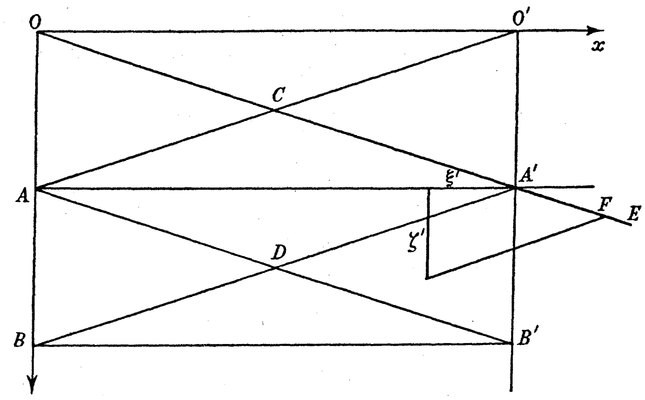

FIGURE 7

In general this expression is somewhat complicated, but fortunately it is only required when $\theta' = \pi$, i.e. along the line $O'O$, which permits considerable simplification.

Some transformation of the expression for G_1 given earlier is desirable. Using the integral representation for $K_0(z)$

$$K_0(z) = \int_1^\infty e^{-z\xi}\frac{d\xi}{\sqrt{(\xi^2-1)}},$$

it is easily seen that

$$K_0[k\sqrt{(r^2+r_0^2-2rr_0\cos\phi)}] = \int_0^\infty \frac{e^{-k\sqrt{[r^2+r_0^2-2rr_0\cos\phi+u^2]}}}{\sqrt{[r^2+r_0^2-2rr_0\cos\phi+u^2]}} du. \tag{41}$$

The second expression, say,

$$B = \frac{1}{\pi}\cos\tfrac{1}{2}\phi\int_0^\infty K_0\{k\sqrt{(r^2+r_0^2+2rr_0\cosh b)}\}\frac{\cosh\frac{1}{2}b\, db}{\cos\phi+\cosh b},$$

can be similarly transformed. We can write

$$B = \frac{1}{2\pi}\cos\tfrac{1}{2}\phi\int_0^\infty \frac{\cosh\frac{1}{2}b}{\cos\phi+\cosh b}\, db\int_0^\infty \exp\left\{-\frac{1}{2}\left[\xi+\frac{k^2(r^2+r_0^2+2rr_0\cosh b)}{\xi}\right]\right\} d\xi,$$

and on inverting the order of integration, the integral with respect to b can be transformed to a more convenient form. If we consider

$$J = \int_0^\infty e^{-a(\cos\phi+\cosh b)} \frac{\cosh\frac{1}{2}b\,db}{\cos\phi+\cosh b},$$

then $\partial J/\partial a$ is easily evaluated

$$\frac{\partial J}{\partial a} = -\sqrt{\left(\frac{\pi}{2a}\right)}\, e^{-2a\cos^2\frac{1}{2}\phi} = \frac{\sqrt{\pi}}{\cos\frac{1}{2}\phi}\frac{\partial K}{\partial a},$$

where $K = \int_{\sqrt{(2a)}\cos\frac{1}{2}\phi}^\infty e^{-t^2}\,dt$. From the above it can be seen that

$$\int_0^\infty \exp\left(\frac{-k^2rr_0\cosh b}{\xi}\right)\frac{\cosh\frac{1}{2}b}{\cos\phi+\cosh b}\,db = \sqrt{\left(\frac{\pi}{2\xi}\right)}\frac{k}{\cos\frac{1}{2}\phi}\exp\left(\frac{k^2rr_0\cos\phi}{\xi}\right)\int_{2\sqrt{(rr_0)}\cos\frac{1}{2}\phi}^\infty \exp\left(\frac{-k^2v^2}{2\xi}\right)dv.$$

Substituting this value in the expression for B, we find

$$B = \frac{k}{2\sqrt{(2\pi)}}\int_0^\infty dv\int_{2\sqrt{(rr_0)}\cos\frac{1}{2}\phi}^\infty \exp\left\{-\frac{1}{2}\left[\xi+\frac{k^2(r^2+r_0^2-2rr_0\cos\phi+v^2)}{\xi}\right]\right\}\frac{d\xi}{\xi^{\frac{3}{2}}}$$

$$= \frac{1}{2}\int_{2\sqrt{(rr_0)}\cos\frac{1}{2}\phi}^\infty \frac{\exp\{-k\sqrt{(r^2+r_0^2-2rr_0\cos\phi+v^2)}\}}{\sqrt{(r^2+r_0^2-2rr_0\cos\phi+v^2)}}\,dv, \quad -\pi<\phi<\pi. \tag{42}$$

These transformations enable us to write the Green's functions G_1, G_2 in somewhat simpler forms. These simpler forms have also the advantage of making it directly obvious that there is no discontinuity in G_1 or G_2 as we go between the regions A, B, C. Thus in the three regions we can write for G_1, in place of (36),

$$G_1(r,\theta;r_0\theta_0) = F(r,r_0,(\theta-\theta_0)) - F\{r,r_0,(\theta+\theta_0)\}, \tag{43}$$

where
$$F(r,r_0,\phi) = \int_0^\infty \frac{\exp\{-k\sqrt{(r^2+r_0^2-2rr_0\cos\phi+t^2)}\}}{\sqrt{(r^2+r_0^2-2rr_0\cos\phi+t^2)}}\,dt$$
$$-\frac{1}{2}\int_{2\sqrt{(rr_0)}\cos\frac{1}{2}\phi}^\infty \frac{\exp\{-k\sqrt{(r^2+r_0^2-2rr_0\cos\phi+t^2)}\}}{\sqrt{(r^2+r_0^2-2rr_0\cos\phi+t^2)}}\,dt.$$

Returning now to the expression for $\bar{\phi}'_2(r'_0,\theta'_0)$, we see this can be written in the form

$$\bar{\phi}'_2(r'_0,\theta'_0) = \frac{1}{2\pi}\int_0^\infty \frac{f(x')}{x'}\{F'(-\theta'_0)-F'(\theta'_0)-F'(2\pi-\theta'_0)+F'(2\pi+\theta'_0)\}\,dx', \tag{44}$$

where for brevity we denote

$$\left\{\frac{\partial}{\partial\phi}F(x',r'_0,\phi)\right\}_{\phi=-\theta_0} = F'(-\theta'_0), \text{ etc.}$$

These results become particularly simple in the case $\theta'_0=\pi$, when the derivatives under the integral signs all vanish, and we are left with

$$F'(\pi) = -F'(-\pi) = \tfrac{1}{2}\sqrt{(x'r'_0)}\,\frac{e^{-\alpha p(x'+r'_0)}}{(x'+r'_0)}. \tag{45}$$

Introducing ξ' in place of r_0' for this case, so that ξ' denotes distance from O' along the span of the aerofoil (figure 7), we have the expression

$$\bar{\phi}_2' = \frac{1}{\pi}\int_0^\infty f(x')\sqrt{\left(\frac{\xi'}{x'}\right)}\frac{e^{-\alpha p(\xi'+x')}}{(\xi'+x')}dx'$$
$$= \frac{2}{\pi\sqrt{(\alpha\pi)}}\frac{\sqrt{\xi'}}{p^{\frac{3}{2}}}\int_0^\infty \frac{x'^{-\frac{1}{2}}\,dx'}{(\xi'+x')}e^{-\alpha p(\xi'+x')}\int_{\sqrt{(x'+2b)}}^\infty e^{-\alpha p\tau^2}\,d\tau \tag{46}$$

on substituting from (38).

We are interested mainly in the resultant pressure on the wing which will be obtained from the wave with transform $\bar{\phi}_2'$, and the corresponding wave $\bar{\phi}_2$ from 0. For this purpose, considering the former wave, we must find $\partial\phi_2'/\partial z$. Remembering that the transform of

$$p^{-\frac{1}{2}}e^{-\alpha p(\tau^2+\xi'+x')} \text{ is } 0,\ z<\alpha(\xi'+x'+\tau^2),$$

$$\frac{1}{\sqrt{[\pi\{z-\alpha(\xi'+x'+\tau^2)\}]}}\ z>\alpha(\xi'+x'+\tau^2),$$

we see that $$\frac{\partial\phi_2'}{\partial z} = \frac{2\alpha^{-\frac{1}{2}}}{\pi^2}\xi'^{\frac{1}{2}}\int_0^{\frac{1}{2}\left\{\frac{z}{\alpha}-(\xi'+2b)\right\}}\frac{dx'}{\xi'+x'}\int_{\sqrt{(x'+2b)}}^{\sqrt{\left\{\frac{z}{\alpha}-(\xi'+x')\right\}}}\frac{d\tau}{\sqrt{\{z-\alpha(\xi'+x')-\alpha\tau^2\}}}, \tag{47}$$

provided $z/\alpha>\xi'+2b$, and is zero if $z/\alpha<\xi'+2b$. As we should expect, this simply means that the wave C_2' produces no effect on the wing until, coming from the leading edge, we cross the line $A'B$ (figure 7). Assuming $z/\alpha>(\xi'+2b)$, we see that

$$\frac{\partial\phi_2'}{\partial z} = \frac{2}{\alpha\pi^2}\sqrt{\xi'}\int_0^{\frac{1}{2}\left\{\frac{z}{\alpha}-(\xi'+2b)\right\}}\frac{\cos^{-1}\sqrt{\{[\alpha(x'+2b)/z-\alpha(\xi'+x')]\}}}{(\xi'+x')}\frac{dx'}{\sqrt{x'}}. \tag{48}$$

We shall denote by ζ' the quantity $(z-2b\alpha)$, so that ξ', ζ' denote the co-ordinates of any point P on the wing, relative to A' (figure 7). The upper limit in the integral over x' simply corresponds to the point F on the line AE, which is the last that can affect conditions at P.

For numerical integration it is convenient to introduce the substitution $x'=\xi'u^2$, giving

$$\frac{\partial\phi_2'}{\partial z} = \frac{4}{\alpha\pi^2}\int_0^{\sqrt{\left\{\frac{1}{2}\left(\frac{\zeta'}{\xi'\alpha}-1\right)\right\}}}\cos^{-1}\sqrt{\left\{\frac{\xi'u^2+2b}{\frac{\zeta'}{\alpha}+2b-\xi'(1+u^2)}\right\}}\frac{du}{1+u^2}. \tag{49}$$

Unfortunately, the integral does not appear to be integrable in simple analytical form. Also it cannot be made to depend on any single parameter, but must ultimately be regarded as a function of the two variables ξ' and ζ'. One point may immediately be noted, however; as we make $\xi'\to 0$,

$$\frac{\partial\phi_2'}{\partial z}\to\frac{2}{\pi\alpha}\cos^{-1}\sqrt{\left(\frac{2b\alpha}{z}\right)},$$

so exactly annulling the pressure discontinuity along the edge $A'B'$ of the wing, which would otherwise appear after A'.

With various other minor substitutions (49) can be brought into a form suitable for systematic computation, and its value has been calculated for $(\xi'/2b)$ and $(\zeta'/2b\alpha)$ lying between 0 and 1. In this way, superposing the plane wave, and the waves C_1, C_1', C_2, C_2', excess pressures have been calculated over a rectangular plate aerofoil span $2b$, and chord $4\alpha b$. The numerical results are discussed in the following section.

The process can be extended indefinitely. The value of ϕ_2' beyond the edge Bz of the aerofoil can first be calculated, and a result of a similar form to (48) then enables the excess pressure from the wave C_3 to be calculated, and so the process can be repeated until the trailing edge of the aerofoil is reached.

Rectangular plate aerofoil at incidence—numerical results

We now consider the numerical results for the rectangular flat-plate aerofoil at incidence. At corresponding points on the upper and lower surface there are, of course, equal and opposite excess pressures due to the disturbance produced in the uniform flow. These excess pressures are found by summing the contributions from the appropriate 'waves' as described earlier—the initial plane wave from the leading edge, supplemented by the cone waves from O, O', A, A', etc. (figure 7). The plane-wave excess pressure will be denoted Δp_0. It is given on the upper surface by

$$\Delta p_0 = -\frac{\epsilon}{\alpha}\rho V^2.$$

We shall use Δp_1, $\Delta p_1'$, Δp_2, $\Delta p_2'$, etc., to denote the excess pressures in the waves from O, O', A, A', etc., respectively.

In tables 1 and 2 are tabulated $\Delta p_1/\Delta p_0$, $\Delta p_2/\Delta p_0$ respectively, calculated from (28) and (49). In table 3 the total excess pressure Δp is tabulated, also as a multiple of Δp_0. The solution is continued only for a wing of chord $4\alpha b$. The possibility of further continuation is discussed above.

Table 1. Values of $\Delta p_1/\Delta p_0$

$z/2b\alpha$ \ $x/2b$	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0	—	—	—	—	—	—	—	—	—	—	—
0·2	1·0	0·5	0·0	—	—	—	—	—	—	—	—
0·4	1·0	0·667	0·5	0·333	0·0	—	—	—	—	—	—
0·6	1·0	0·732	0·608	0·500	0·392	0·268	0·0	—	—	—	—
0·8	1·0	0·769	0·667	0·580	0·500	0·420	0·333	0·231	0·0	—	—
1·0	1·0	0·795	0·704	0·631	0·564	0·500	0·436	0·369	0·296	0·205	0·0
1·2	1·0	0·814	0·732	0·667	0·608	0·553	0·500	0·447	0·392	0·333	0·268
1·4	1·0	0·828	0·753	0·694	0·640	0·592	0·546	0·500	0·454	0·408	0·360
1·6	1·0	0·839	0·769	0·714	0·667	0·623	0·580	0·540	0·500	0·460	0·420
1·8	1·0	0·849	0·784	0·734	0·687	0·646	0·608	0·571	0·536	0·500	0·464
2·0	1·0	0·856	0·795	0·747	0·704	0·667	0·631	0·597	0·564	0·532	0·500

Table 2. Values of $\Delta p_2/\Delta p_0$

$\xi'/2b\alpha$ \ $\xi'/2b$	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·0	0·0	—	—	—	—	—	—	—	—	—	—
0·2	0·268	0·062	0·0	—	—	—	—	—	—	—	—
0·4	0·358	0·150	0·0845	0·038	0·0	—	—	—	—	—	—
0·6	0·419	0·215	0·148	0·101	0·625	0·030	0·0	—	—	—	—
0·8	0·464	0·264	0·198	0·150	0·113	0·081	0·052	0·025	0·0	—	—
1·0	0·500	0·306	0·239	0·196	0·155	0·124	0·095	0·070	0·045	0·022	0·0

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TABLE 3. PRESSURE ON RECTANGULAR AEROFOIL AT INCIDENCE (Δp_0 = TWO-DIMENSIONAL PRESSURE). VALUES OF $\Delta p/\Delta p_0$

$z/2b\alpha$ \ $x/2b$	0	0·1	0·2	0·3	0·4	0·5
0	0	1·0	1·0	1·0	1·0	1·0
0·2	0	0·5	1·0	1·0	1·0	1·0
0·4	0	0·333	0·5	0·667	1·0	1·0
0·6	0	0·268	0·392	0·500	0·608	0·464
0·8	0	0·233	0·333	0·189	0·167	0·160
1·0	0	0	0	0	0	0
1·2	0	−0·085	−0·123	−0·114	−0·108	−0·106
1·4	0	−0·086	−0·122	−0·156	−0·185	−0·184
1·6	0	−0·085	−0·122	−0·155	−0·184	−0·184
1·8	0	−0·085	−0·122	−0·128	−0·130	−0·130
2·0	0	−0·060	−0·075	−0·078	−0·085	−0·082

The general distribution of the excess pressure is clear from table 3. The two-dimensional value Δp_0 is steadily eaten into by the waves Δp_1, $\Delta p_1'$ radiating from O and O'. By the time line AA' is reached the excess pressure has fallen to zero all along the span. Thereafter the excess pressure changes sign, and becomes steadily more negative with increasing z, until the arrival of the waves from A and A' reverses the process once more. Physically, the behaviour seems quite clear when the hydrodynamic analogue of the waves produced by transverse movement of a plank is considered. Excess pressure on the aerofoil is now replaced by the height of the water surface along the plank. As the plank is initially jerked into motion the level piles up on one side and sinks on the other, but waves running round the edges of the plank gradually rectify the difference of level at the plank, until as we have seen at time $t=(2b/c)$ the level has sunk to zero all round the plank. Thereafter the level tends to sink farther on the leading side of the plank and to pile up on the other, but again this tendency is counteracted by the new waves running in from the edges. So, in fact, we have an oscillatory process—the oscillations in level becoming smaller, and the motion settling down to a steady one, as the wave energy spreads out from the plank over the surface of the water.

As pointed out earlier, the waves coming in at A, A' ensure that there is no discontinuity of pressure at the edge of the aerofoil (or of water level round the edge of the plank). The detailed results given are broken off at $z=4\alpha b$. If we continued the solution we should find the excess pressure falling to zero again, along the backward Mach lines from B and B', then once more rising to smaller positive values.

It may be well before discussing lift coefficients and the position of the centre of pressure for a rectangular plate at incidence to recall the limitations of the linearized supersonic aerofoil theory. A good discussion of these is given by Lighthill for the two-dimensional case. From this it is clear that the range of applicability of the linearized theory is quite restricted. It underestimates the pressure behind a shock wave corresponding to compressive deflexion round a corner, and by a comparable amount overestimates the suction produced by an expansion through the same angle. These effects tend to cancel out, in evaluating the total lift or drag on any aerofoil, but in general will lead to a wrong estimation of moments and of the position of the centre of pressure. The second-order two-dimensional theory of Busemann gives much better agreement with the exact shock wave and expansion calculations, though still erring on the same side as the linearized theory. This second-order theory expresses the excess pressure at any point on a two-dimensional aerofoil in the form

$$\Delta p = A\eta + B\eta^2,$$

where η denotes the local slope of the aerofoil. For a flat plate the centre of pressure will thus still remain at half-chord, but for a practical aerofoil, with thickness at incidence, the centre of pressure will move forward. Details are given by Lighthill.

A further limitation of the two-dimensional linearized theory occurs at Mach numbers so low that the shock wave at the leading edge becomes detached. At such Mach numbers the whole basis of the theory has broken down, and, indeed, no theoretical method to date offers much promise.

It remains, however, that at moderate Mach numbers and angles of incidence the linearized theory provides a reasonable guide to the actual behaviour, and one whose accuracy can be easily estimated by comparison with the exact two-dimensional calculations. When we extend to the three-dimensional case the physical limitations on the linearized theory must remain very similar. Unfortunately, a second-order theory is no longer simple to find, but the type of correction to be applied to the linearized theory can be roughly deduced from those in the two-dimensional case.

For the finite rectangular plate at incidence the lift can easily be calculated for aerofoils with chord less than $2b\alpha$ (figure 7) simply by adding the contributions from ϕ_0, ϕ_1 and ϕ_1'. If A is used to denote the aspect ratio we find for such aerofoils for the total lift L

$$L = L_0\left(1 - \frac{1}{2A\alpha}\right), \quad \text{provided } A > \frac{1}{\alpha},$$

where L_0 is used to denote the lift force derived using the two-dimensional value for the excess pressure. With similar restriction on A the distance of the centre of pressure ahead of half-chord can also be calculated. It is given by $\frac{c}{6}\left(\frac{1}{2\alpha A - 1}\right)$, where, as usual, c denotes the chord. These results are given by Taunt & Ward (1946). The value of the formula for the centre of pressure is doubtful, as noted also by Lighthill, for practical purposes, as for a reasonably thick aerofoil the second-order correction in the two-dimensional theory shifts the centre of pressure forward by an amount comparable with that given here. Taunt & Ward suggest a rough method of applying the Busemann correction to the three-dimensional theory.

For aspect ratios greater than $1/\alpha$ the above results break down, and there now appears no simple analytical formula for the lift and moments. They can, however, be quite simply calculated by numerical integration from table 3. For example, for $A = 1/2\alpha$ it is found that $L \doteqdot 1/5L_0$.

Symmetrical rectangular aerofoil at zero incidence

From the earlier considerations of the boundary conditions it is clear that the linearized treatment of any symmetrical aerofoil, of rectangular plan at any incidence, may be found by compounding the incidence case for a flat plate, dealt with in the previous section, with the treatment of the aerofoil shape at zero incidence. A general method for handling this latter problem has already been given. However, it may be worth while reconsidering the problem by the methods of the last section.

As a first simple case we consider a supersonic stream impinging symmetrically against a semi-infinite wedge of semi-angle ϵ say. In plan, we assume the leading edge to be situated along the x-axis and the tip lies along the z-axis. The Laplace transform of the velocity potential again satisfies the equation

$$\frac{\partial^2\overline{\phi}}{\partial x^2}+\frac{\partial^2\overline{\phi}}{\partial y^2}=\alpha^2p^2\overline{\phi},$$

together with the boundary condition

$$\frac{\partial\overline{\phi}}{\partial y}=\frac{\epsilon}{p} \quad \text{when} \quad y=0+,\ x>0$$

$$=-\frac{\epsilon}{p} \quad \text{when} \quad y=0-,\ x>0.$$

The value of $\overline{\phi}$ can at once be written down from the Green's function formulae, (31) and (36),

$$\overline{\phi}(r_0,\theta_0)=-\frac{\epsilon}{p\pi}\int_0^\infty K_0\{\alpha p\sqrt{(r^2+r_0^2-2rr_0\cos\theta_0)}\}\,dr, \tag{50}$$

where, as usual, (r_0,θ_0) are polar co-ordinates in the (x,y) plane.

The interpretation of (50) presents no difficulties, and will not be given in detail. We find that the excess pressure Δp at any point (x,z) on the surface (within the Mach cone from the leading corner) can be expressed

$$\Delta p=+\rho U^2\frac{\epsilon}{\pi\alpha}\left(\pi-\cos^{-1}\frac{\alpha x}{z}\right)=\Delta p_0\left(1-\frac{1}{\pi}\cos^{-1}\frac{\alpha x}{z}\right), \tag{51}$$

where Δp_0 represents the two-dimensional value. More generally at any point co-ordinates (x,y,z), $x>0$

$$\Delta p(x,y,z)=\Delta p_0\left(1-\frac{1}{\pi}\cos^{-1}\frac{\alpha x}{\sqrt{(z^2-\alpha^2y^2)}}\right). \tag{52}$$

This excess pressure can, as in the case of the flat plate at incidence, be regarded as made up of a plane-wave contribution together with a cone wave radiating from the corner O. There is an important distinction now, however, in that the excess pressure is symmetrical on the two sides of the aerofoil. As a consequence, in treating the case of the wing of finite span, $2b$ say, no further waves need be added at the points A, A', etc. (figure 7). The complete solution for all values of z is given by the plane wave together with the two cone waves from O, O', for these suffice to satisfy the boundary conditions and preserve the continuity of the velocity components throughout. We may refer again to the water-wave analogy. The two sides of the plank are now moved apart with a given velocity, and initially the water piles up on either side. The equalizing waves from the corners O, O' are now the same on either side; thus there are no reflected waves produced when, for example, the wave from O runs off the plank at O'.

Thick aerofoils may be treated by superposition of solutions of the type given above, or if their surface is of suitable analytical form it may be more convenient to substitute this directly in deriving the boundary condition for $\partial\bar{\phi}/\partial y$. For example, to take the simplest case, the double-wedge aerofoil is simulated by subtracting from the solution for a symmetrical wedge, semi-angle ϵ, vertex at $z=0$, that for a similar wedge semi-angle 2ϵ, vertex at $z=\frac{1}{2}c$. More generally, if we consider a semi-infinite aerofoil, with cross-section given by

$$y=\eta(z),$$

the pressure at a point on its surface within the Mach triangle from the leading corner O can be written in the form

$$\frac{\Delta p-\Delta p_0}{\rho U^2}=-\frac{\eta'(0)}{\pi\alpha}\cos^{-1}\left(\frac{\alpha x}{z}\right)-\frac{1}{\pi\alpha}\int_0^{z-\alpha x}\cos^{-1}\frac{\alpha x}{z-\zeta}\eta''(\zeta)\,d\zeta, \tag{53}$$

where Δp_0 is used to denote the two-dimensional value of the excess pressure.

It is easy to demonstrate from (53), as was shown by Lighthill by other means, that, for symmetrical rectangular wings of aspect ratio greater than $1/\alpha$, there is no induced drag at zero incidence. If the aerofoil is of chord c, then the decrement of drag due to the pressure defect (53) may be written

$$\Delta D=\int_0^c\eta'(z)\,dz\int_0^{z\tan\mu}\frac{1}{\pi\alpha}\left\{\eta'(0)\cos^{-1}\frac{\alpha x}{z}+\int_0^{z-\alpha x}\cos^{-1}\frac{\alpha x}{z-\zeta}\eta''(\zeta)\,d\zeta\right\}dx$$
$$=\frac{1}{\pi\alpha}\int_0^c\eta'(z)\,dz\left\{\int_0^{z/\alpha}\eta'(0)\cos^{-1}\frac{\alpha x}{z}\,dx+\int_0^z\eta''(\zeta)\,d\zeta\int_0^{(z-\zeta)/\alpha}\cos^{-1}\frac{\alpha x}{z-\zeta}\,dx\right\}.$$

Introducing $\alpha x/z=\sigma$ in the first integral, and $\alpha x/(z-\zeta)=\sigma$ in the second, we find

$$\Delta D=\frac{1}{\pi\alpha^2}\int_0^1\cos^{-1}\sigma\,d\sigma\left[\int_0^c\eta'(z)\,dz\left\{\eta'(0)+\int_0^z\eta''(\zeta)\,(z-\zeta)\,d\zeta\right\}\right]$$
$$=\frac{1}{\pi\alpha^2}\int_0^1\cos^{-1}\sigma\,d\sigma\left\{\int_0^c\eta'(z)\,dz\right\}^2=0.$$

The result depends essentially on the fact that the excess pressure given in (51) is of the cone-field type depending not on the distance of the point of measurement from the corner, but on its angular position with reference to it. It will cease to hold when the aspect ratio is less than $1/\alpha$. Some calculations have been made for this case, and it is found, for example, that for a double-wedge aerofoil, aspect ratio $1/2\alpha$, the drag is about two-thirds of that given by the two-dimensional theory.

Rectangular aerofoils—more general problems

So far we have been considering, for the most part, problems of the 'cone-field' type—the only exception being the prolongation of the solution for the flat plate at incidence. The methods developed, however, permit the solution of more general problems. Before leaving the case of rectangular plan aerofoils with leading edge normal to and tips parallel to the stream, it may be worth while to illustrate roughly some more general cases of this type.

(a) Tapered aerofoil

We consider a symmetrical aerofoil of span $2b$ at incidence ϵ, with thickness tapered across the span according to a sine factor, so that apart from the incidence, the upper surface may be represented by

$$y = \eta(z)\sin\frac{\pi x}{2b}. \tag{54}$$

The aerofoil has thus zero thickness at either tip. As usual the solution is found by superposition of the symmetrical zero incidence case, requiring

$$\frac{\partial\phi}{\partial y} = \pm\eta'(z)\frac{\sin\pi x}{2b} \quad \text{as} \quad y\to\pm 0,\ 0<x<2b,$$

and of the anti-symmetrical case

$$\frac{\partial\phi}{\partial y}\to -\epsilon\frac{\sin\pi x}{2b} \quad \text{as} \quad y\to\pm 0,\ 0<x<2b.$$

The symmetrical problem, giving the wave drag at zero incidence, is of no special interest, and can be solved by the methods given earlier. The anti-symmetrical problem is of a new type, but, at least for aerofoils of aspect ratio greater than $2/\alpha$ can be solved fairly directly by the Green's function methods, using the function G_2, (31), (36) and (43). For the transform $\bar\phi$ we find

$$\bar\phi = \frac{\epsilon}{\pi p}\int_0^\infty \sin\frac{\pi r}{2b}\,dr\int_0^{2\sqrt{(rr_0)}\cos\frac{1}{2}\theta_0}\frac{\exp\left[-\alpha p\sqrt{(r^2+r_0^2-2rr_0\cos\theta_0+u^2)}\right]}{\sqrt{(r^2+r_0^2-2rr_0\cos\theta_0+u^2)}}\,du, \tag{55}$$

where the continuation of the boundary condition for $\partial\phi/\partial y$ beyond $x = 2b$ will not affect conditions in the range $0\leqslant x\leqslant b$ for the aspect ratios considered. Interpreting formally, introducing Dirac's δ-function, we have for $\partial\phi/\partial z$, in the case $\theta_0 = 0$,

$$\frac{\partial\phi}{\partial z} = +\frac{\epsilon}{\pi}\iint\frac{\delta(z-\alpha R)}{R}\sin\frac{\pi r}{2b}\,dr\,du, \tag{56}$$

where the integral is taken over the area between the positive r-axis, and the upper half of the parabola $u^2 = 4rr_0$ in the (u, r) plane. Here we have written

$$R^2 = (r-r_0)^2+u^2.$$

Introducing polar co-ordinates R, θ in the r, u plane, we may write

$$\frac{\partial\phi}{\partial z} = \frac{\epsilon}{\pi}\iint\delta(z-\alpha R)\sin\frac{\pi(r_0+R\cos\theta)}{2b}\,dR\,d\theta,$$

where θ is measured from the r-axis, the integral being over the same area as previously. From the properties of the δ-function it is clear that significant contributions to the integral come only from the area around the circumference of the circle $(r-r_0)^2+u^2 = z^2/\alpha^2$. If $z<\alpha r_0$ the whole of this circle is contained within the parabola, and we find

$$\begin{aligned}\frac{\partial\phi}{\partial z} &= +\frac{\epsilon}{\pi\alpha}\int_0^\pi \sin\left\{\frac{\pi(r_0+z/\alpha\cos\theta)}{2b}\right\}d\theta\\ &= \frac{\epsilon}{\alpha}\sin\frac{\pi r_0}{2b}J_0\left(\frac{\pi z}{2b\alpha}\right).\end{aligned} \tag{57}$$

If, on the other hand, $z > \alpha r_0$, i.e. the point considered is within the Mach triangle of the corner O, then

$$\frac{\partial \phi}{\partial z} = \frac{\epsilon}{\pi\alpha}\int_0^{\theta_1} \sin\left\{\frac{\pi}{2b}\left(r_0 + \frac{z}{\alpha}\cos\theta\right)\right\} d\theta, \quad \text{where} \quad \sin\frac{\theta_1}{2} = \sqrt{\frac{r_0\alpha}{z}}. \tag{58}$$

The method can be extended immediately to any other tapering function. In particular, when the cross-section is constant we find $\partial\phi/\partial z = \epsilon\theta_1/\pi\alpha$, confirming the result obtained earlier.

(b) *Aerofoil with curved leading edge—wing tip parallel to stream.*

As a last example of aerofoils with wing tips parallel to the stream we now consider the case of a curved leading edge with the restriction that the tangent to the leading edge must at no point be swept back beyond the Mach angle. The plan of the aerofoil may be as depicted in figure 8. We shall again restrict the argument to cases where the Mach lines

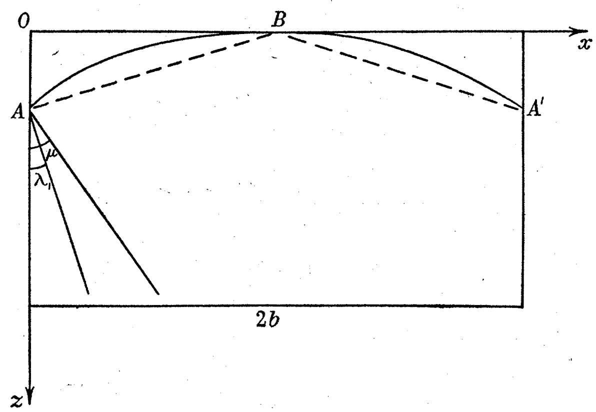

FIGURE 8

from the corners A, A' do not intersect on the aerofoil surface. A symmetrical wing at zero incidence presents no special problems. If we consider the wing as a flat plate at incidence ϵ, then by the restriction on the leading edge's sweep back it is ensured that there is no disturbance of the stream ahead of the leading edge. Accordingly, if we represent the leading edge by the curve

$$z = \zeta(x) \quad (x > 0),$$

then we can form

$$\frac{\partial \bar{\phi}}{\partial y} = \int_0^\infty e^{-pz}\frac{\partial \phi}{\partial y}\,dz = -\frac{\epsilon}{p}e^{-p\zeta(x)}. \tag{59}$$

We may assume the curve $z = \zeta(x)$ continued beyond $x = 2b$ in any convenient way, as this will not alter conditions at points on the aerofoil with which we shall be concerned, such that $0 \leqslant x \leqslant b$. Now applying the Green's function method we have the solution for $\bar{\phi}$ along the x-axis, at the point x_0,

$$\bar{\phi} = -\frac{\epsilon}{p\pi}\int_0^\infty e^{-p\zeta(x)}\,dx\int_0^{2\sqrt{(xx_0)}} \frac{\exp\left[-\alpha p\sqrt{\{(x-x_0)^2+u^2\}}\right]}{\sqrt{\{(x-x_0)^2+u^2\}}}\,du. \tag{60}$$

Interpreting this result, as in the last section, and once more introducing polar co-ordinates R, θ in the (x, u) plane, with pole at $(x_0, 0)$, we find for $(\partial\phi/\partial z)$ at the point (x_0, z) on the plate

$$\frac{\partial \phi}{\partial z} = -\frac{\epsilon}{\pi}\iint_S \partial\{z - \alpha R - \zeta(x)\}\,dR\,d\theta, \tag{61}$$

where the area S, as previously, denotes the upper half of the parabola $u^2 = 4xx_0$. The integral with respect to R can be simply carried out. We find

$$\frac{\partial\phi}{\partial z} = \frac{\epsilon}{\pi\alpha}\int_0^{\theta_1}\frac{d\theta}{1+\zeta'(x)/\alpha\cos\theta}, \tag{62}$$

where $\theta = \theta_1$ corresponds to the intersection between the curves $u^2 = 4xx_0$, and $z/\alpha = R+\zeta(x)$. Also in the denominator of the integrand x is a function of θ, given by

$$z = \alpha R+\zeta(x) = \frac{\alpha(x-x_0)}{\cos\theta}+\zeta(x).$$

Exact integration of (62) is not always possible, but the formula is not a very troublesome one for numerical integration. Some simple cases are directly integrable; e.g. the case, treated by Ward by the cone-field method, where the leading edge consists of the straight lines, shown dotted in figure 8. In this case we may write

$$\zeta(x) = h\left(1-\frac{x}{b}\right) \quad (0<x<b)$$

$$= h\left(\frac{x}{b}-1\right) \quad (b<x<2b).$$

Hence $\zeta'(x) = \mp m$ according as $0<x<b$ or $b<x<2b$, where $m = h/b$.

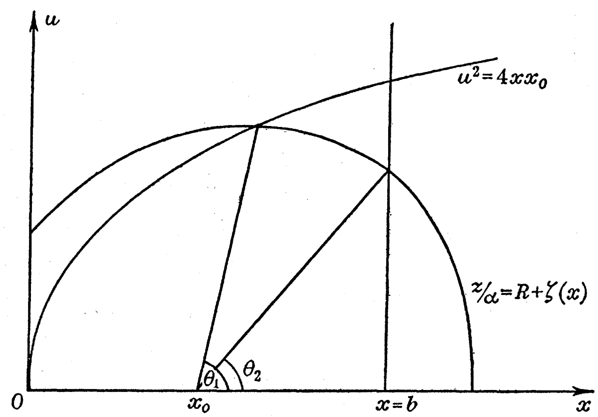

FIGURE 9

Various possibilities can now arise, supposing that x_0 is to be restricted to be less than b. First, if $z<(\alpha x_0+h)$ and $<\alpha(b-x_0)$, then the curve $z/\alpha = R+\zeta(x)$ lies entirely within the parabola and

$$\frac{\partial\phi}{\partial z} = \frac{\epsilon}{\pi}\int_0^{\pi}\frac{d\theta}{\alpha-m\cos\theta} = \frac{\epsilon}{\surd(\alpha^2-m^2)}. \tag{63}$$

This is the two-dimensional value for the swept-back wing. The restriction on z simply means that the point lies outside the Mach cones from both A and B.

Secondly, suppose $z>\alpha x_0+h$, but $<\alpha(b-x_0)$. The point lies within the Mach cone from A. Then

$$\frac{\partial\phi}{\partial z} = \frac{\epsilon}{\pi}\int_0^{\theta_1}\frac{d\theta}{\alpha-m\cos\theta} = \frac{2\epsilon}{\pi\surd(\alpha^2-m^2)}\tan^{-1}\left\{\sqrt{\left(\frac{\alpha+m}{\alpha-m}\right)}\tan\frac{\theta_1}{2}\right\}, \tag{64}$$

where θ_1, corresponding to the intersection of the two curves in figure 9, is such that

$$\tan^2\frac{\theta_1}{2} = \frac{x_0(\alpha-m)}{z-h-\alpha x_0}.$$

If we write $$\sigma_1 = \frac{\tan\lambda_1}{\tan\mu} = \frac{\alpha x_0}{(z-h)} \quad \text{(figure 8)},$$

then (64) is easily seen to be equivalent to Ward's result,

$$\frac{\partial\phi}{\partial z} = \frac{2\epsilon}{\pi\sqrt{(\alpha^2-m^2)}}\sin^{-1}\sqrt{\left\{\frac{(\alpha+m)\,\sigma_1}{(\alpha+m\sigma_1)}\right\}}. \tag{65}$$

Finally, if $z>\alpha(b-x_0)$, but $<\alpha x_0+h$, so that the point is within the Mach cone from B, then, with the notation of figure 9,

$$\frac{\partial\phi}{\partial z} = \frac{\epsilon}{\pi}\left\{\int_0^{\theta_2}\frac{d\theta}{\alpha+m\cos\theta} + \int_{\theta_2}^{\pi}\frac{d\theta}{\alpha-m\cos\theta}\right\}.$$

This is easily evaluated to be

$$\frac{\partial\phi}{\partial z} = \frac{2\epsilon}{\pi\sqrt{(\alpha^2-m^2)}}\cot^{-1}\left\{\frac{m\sin\theta_2}{\sqrt{(\alpha^2-m^2)}}\right\}, \tag{66}$$

where $\cos\theta_2 = \dfrac{(b-x_0)\,\alpha}{z} = \sigma_2$ say. An equivalent result was given by Ward.

Conclusions

So far this argument has been confined exclusively to aerofoils with wing tips parallel to the stream. The case of zero incidence on a symmetrical aerofoil has proved to be susceptible of a general solution not confined to such cases. In actual problems the methods depending on the use of the appropriate Green's function for a semi-infinite barrier are found to be the most convenient way of arriving at a solution. The case of non-zero incidence is more difficult of treatment. However, using operational methods, based on the classical solutions of diffraction problems by Sommerfeld and others, a wide range of problems has been seen to be soluble in a systematic manner. These include all problems of the type considered, previously solved by the supersonic source and cone-field methods, and in addition can be generalized to other problems not amenable to these more specialized techniques.

The operational methods, with their wave interpretation and suggestive analogies, undoubtedly form an effective method of dealing with the linearized equation for supersonic flow. It is also satisfying, if scarcely surprising, to note the formal analogy between the diffraction and aerofoil problems. It is the author's opinion that the use of 'sources' and 'doublets' as a technique in the solution of supersonic problems is really misplaced, and a remnant from subsonic theory. They may, of course, be of physical interest in the interpretation of solutions otherwise obtained.

The outstanding problems still to be dealt with by the operational methods are those where the wing tips are no longer parallel to the stream, and cases with the leading edge swept back beyond the Mach angle. A variety of such problems are considered by the author in Part II.